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Weyl's rule and Wigner equivalents for phase space multinomials

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Abstract. Weyl's rule of association is applied to the multinomial basis set for classical phase space functions and, using an appropriate symplectic notation, the corresponding basis set in quantum-mechanical phase space is found to consist of averages of multinomial operators. Groenewold's rule for the Wigner equivalent of a product of two operators is generalised to an arbitrary number of multipliers. An explicit expansion is derived for general multinomial operators in terms of symmetric multinomials (i.e. the basis set averages) and powers of $\hbar$. This makes it simple to apply Weyl's correspondence rule to both classical and quantum-mechanical functions with multinomial expansions.

1. Introduction

Weyl's rule of association (Weyl 1931, see also Cohen 1966, § 2) provides a one-one linear correspondence between quantum-mechanical phase space operators and classical functions of the position and momentum coordinates. In one dimension, it is usually expressed as a mapping between the basis sets $C := \{e(\theta q + \tau p)\}$ and $Q :=$ $\{e(\theta\hat{q} + \tau\hat{p})$, where $e(x) := \mathrm{e}^{\mathrm{i}x}$ and θ, τ range over the real numbers.

Although Weyl was postulating only a one-way mapping $C \to Q$, his rule is in fact one-one (Imre *et al* 1967, § II). It is given by

$$e(\theta q + \tau p) \leftrightarrow e(\theta\hat{q} + \tau\hat{p}). \tag{1.1}$$

The classical phase space function induced by Weyl's rule from the operator function $\hat{A}$ is denoted by either $(\hat{A})_{\mathrm{W}}$ or A_{W}. Thus, if $A_{\mathrm{W}}(q, p)$ has a Fourier transform $\alpha(\theta, \tau)$ such that

$$A_{\mathrm{W}}(q, p) = \int \alpha(\theta, \tau) e(\theta q + \tau p)\, \mathrm{d}\theta\, \mathrm{d}\tau,$$

then equation (1.1) implies that

$$\hat{A}(\hat{q}, \hat{p}) = \int \alpha(\theta, \tau) e(\theta\hat{q} + \tau\hat{p})\, \mathrm{d}\theta\, \mathrm{d}\tau.$$

Use of the subscript 'W' is politic; A_{W} may be referred to as the 'Weyl transform' of $\hat{A}$ (e.g. Leaf 1968) or as the 'Wigner equivalent' of $\hat{A}$ (e.g. Imre *et al* 1967). The

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latter term emphasises the close link between Weyl's rule and the Wigner phase space distribution function $W(q, p, t)$ (Wigner 1932) which is such that the expectation value of the operator $\hat{A}$ is given by

$$\langle \hat{A} \rangle = \int A_{\mathrm{W}}(q, p)\, W(q, p, t)\, \mathrm{d}q\, \mathrm{d}p.$$

This paper intends to illuminate Weyl's rule and make it more easily applicable in many cases, by examining the correspondence in relation to the basis set of multinomials in classical phase space. In one dimension this is the set $C' := \{q^m p^n\}$ where m, n are non-negative integers. McCoy (1932) showed that the rule of association in this case takes the form

$$q^m p^n \leftrightarrow \frac{1}{2^m} \sum_{r=0}^{m} \binom{m}{r} \hat{q}^r \hat{p}^n \hat{q}^{m-r}, \tag{1.2a}$$

$$= \frac{1}{2^n} \sum_{s=0}^{n} \binom{n}{s} \hat{p}^s \hat{q}^m \hat{p}^{n-s}. \tag{1.2b}$$

His proof was complicated; a simpler method was outlined by Mukunda (1979, § IV) as follows: in equation (1.1) rewrite $e(\theta\hat{q} + \tau\hat{p})$ as $e(\frac{1}{2}\theta\hat{q})e(\tau\hat{p})e(\frac{1}{2}\theta\hat{q})$ (the result given in the appendix may be used to show this). The exponential functions are now expanded and coefficients of $\theta^m \tau^n$ equated to give result (1.2a). Equation (1.2b) may be obtained in a similar manner.

Both forms of Weyl's rule, (1.1) and (1.2) above, are easily applied in the direction classical $\rightarrow$ quantum-mechanical. It is not such a straightforward task, however, to proceed the other way; how are general operator functions expressed in terms of either of the basis sets Q or $Q' := \{2^{-m} \Sigma_{r=0}^{m} \binom{m}{r} \hat{q}^r \hat{p}^n \hat{q}^{m-r}\}$? For Q the result $e(\theta\hat{q} + \tau\hat{p}) = e(\frac{1}{2}\theta\tau\hbar)e(\theta\hat{q})e(\tau\hat{p})$ is used; if $\hat{A}$ is now expressed in normal form, i.e. the $\hat{q}$ factors precede the $\hat{p}$ factors, then A_{W} is easily calculated (see Cohen 1966, § 2).

In the case of Q', it is messy to find the normal form of the general multinomial operator $\hat{q}^{m_1} \hat{p}^{n_1} \ldots \hat{q}^{m_r} \hat{p}^{n_r}$. However, it will be found that significant formal simplicity is achieved by combining the position and momentum coordinates into one vector and the normal form may be bypassed altogether in determining the Wigner equivalents of multinomial operators. This notation is introduced in the next section.

Wigner equivalents of operator products are studied in the third section and applied to phase space multinomials in § 4. Where Q' is generalised to a basis set for phase space of arbitrary even dimension, an explicit expansion of multinomials in terms of the basis elements is found. Finally, a discussion is given in § 5.

2. Symplectic form of Weyl's rule

Weyl's rule is easily generalised to N dimensions; in equation (1.1) replace q, p, $\hat{q}$, $\hat{p}$, θ, τ by N-vectors and multiplication by the scalar product. A simpler form is obtained, however, using symplectic notation (see e.g. Goldstein 1980 and Andrews 1983, § 2).

Define the phase space vector x by $x^i := q_i$, $x^{N+i} := p_i$ for $i = 1, 2, \ldots, N$. The operator $\hat{x}$ is defined in an analogous fashion. The rule of association (1.1) then generalises to

$$e(\theta^\alpha x^\alpha) \leftrightarrow e(\theta^\alpha \hat{x}^\alpha), \tag{2.1}$$

where $\boldsymbol{\theta}$ is a $2N$-vector with components θ^α (Greek indices run from 1 to $2N$ and repeated indices are to be summed over).

It is also convenient to define the $2N \times 2N$ block matrix ε by

$$\varepsilon = \left(\begin{array}{c|c} 0 & I_N \\ \hline -I_N & 0 \end{array}\right), \tag{2.2}$$

where I_N is the $N \times N$ identity matrix. The basic commutation relations $[\hat{q}_i, \hat{p}_j] = \mathrm{i}\hbar\delta_{ij}$ now take the form

$$[\hat{x}^\alpha, \hat{x}^\beta] = \mathrm{i}\hbar\varepsilon^{\alpha\beta}, \tag{2.3}$$

where $\varepsilon^{\alpha\beta}$ is the (α, β) element of ε.

The generalisation of equations (1.2) may be found by expanding the exponentials in the correspondence relation (2.1) and equating the coefficients of $\theta^{\alpha_1} \ldots \theta^{\alpha_n}$:

$$\sum_{m=0}^{\infty} \frac{\mathrm{i}^m}{m!}(\theta^\beta x^\beta)^m \leftrightarrow \sum_{n=0}^{\infty} \frac{\mathrm{i}^n}{n!}(\theta^\beta \hat{x}^\beta)^n$$

i.e.,

$$\sum_{m=0}^{\infty} \frac{\mathrm{i}^m}{m!}\theta^{\beta_1} \ldots \theta^{\beta_m} x^{\beta_1} \ldots x^{\beta_m} \leftrightarrow \sum_{n=0}^{\infty} \frac{\mathrm{i}^n}{n!}\theta^{\beta_1} \ldots \theta^{\beta_n} \hat{x}^{\beta_1} \ldots \hat{x}^{\beta_n}.$$

Now $\theta^{\beta_1} \ldots \theta^{\beta_m} \equiv \theta^{\alpha_1} \ldots \theta^{\alpha_n}$ if and only if $m = n$ and $\beta_1, \ldots, \beta_n = \alpha_{i_1}, \ldots, \alpha_{i_n}$, where $i_1, \ldots, i_n$ is any permutation P of $1, \ldots, n$. Hence $n! x^{\alpha_1} \ldots x^{\alpha_n} \leftrightarrow \sum_P \hat{x}^{\alpha_{i_1}} \ldots \hat{x}^{\alpha_{i_n}}$, or

$$x^{\alpha_1} \ldots x^{\alpha_n} \leftrightarrow \hat{x}^{(\alpha_1} \ldots \hat{x}^{\alpha_n)} \tag{2.4}$$

where ( ) are the usual symmetrising brackets. Thus equations (1.2) have an elegant generalisation: the operator corresponding to the multinomial $x^{\alpha_1} \ldots x^{\alpha_n}$ is the *average* of all the possible permutations $\hat{x}^{\alpha_{i_1}} \ldots \hat{x}^{\alpha_{i_n}}$.

It can be seen from (2.4) that Weyl's rule may be written as a simple correspondence between the basis sets $\{x^{\alpha_1} \ldots x^{\alpha_n}\}$ and $\{\hat{x}^{(\alpha_1} \ldots \hat{x}^{\alpha_n)}\}$. A method of expressing the operator $\hat{x}^{\alpha_1} \ldots \hat{x}^{\alpha_n}$ in terms of symmetric multinomials remains to be found; this is done in § 4.

3. Wigner equivalents of products

In general, the operators $\hat{A}$, $\hat{B}$ do not commute and hence one cannot have $(\hat{A}\hat{B})_\mathrm{W} \equiv A_\mathrm{W} B_\mathrm{W}$, since the correspondence is one–one and linear. However, a similar equation does hold where the right-hand side is multiplied by a differential operator. This is called Groenewold's rule (Groenewold 1946; see Imre *et al* 1967, § III) and will be shown here to be a special case of a more general rule which gives the Wigner equivalent of a general product of operators.

Suppose $\hat{A}$ has the form

$$\hat{A}(\hat{\boldsymbol{x}}) = \hat{A}^1(\hat{\boldsymbol{x}}) \ldots \hat{A}^n(\hat{\boldsymbol{x}})$$

and

$$\hat{A}^i(\hat{\boldsymbol{x}}) = \int \alpha_i(\boldsymbol{\theta}) e(\theta^\alpha \hat{x}^\alpha)\, \mathrm{d}\boldsymbol{\theta} \qquad \text{for } i = 1, \ldots, n.$$

Now define $\alpha(\Theta) := \alpha_1(\boldsymbol{\theta}_1) \ldots \alpha_n(\boldsymbol{\theta}_n)$, where $\Theta := (\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_n)$ is a $2nN$-vector. Then

$$\hat{A}(\hat{x}) = \int \alpha(\Theta) e(\theta_1{}^\alpha \hat{x}^\alpha) \ldots e(\theta_n^\alpha \hat{x}^\alpha)\, \mathrm{d}\Theta$$

$$= \int \alpha(\Theta) e\left(\tfrac{1}{2}\left\{\sum_{j<k} (i\theta_j{}^\alpha \hat{x}^\alpha, i\theta_k{}^\beta \hat{x}^\beta)\right\}\right) e\left(\hat{x}^\alpha \sum_j \theta_j{}^\alpha\right) \mathrm{d}\Theta$$

using the result given in the appendix, where j, k range over $1, \ldots, n$. Hence

$$\hat{A}(x) = \int \alpha(\Theta) e\left(-\tfrac{1}{2}\hbar \sum_{j<k} \varepsilon^{\alpha\beta} \theta_j{}^\alpha \theta_k{}^\beta\right) e\left(\hat{x}^\alpha \sum_j \theta_j{}^\alpha\right) \mathrm{d}\Theta$$

from equation (2.3).

Weyl's rule in the symplectic form (2.1) may now be applied to give

$$A_{\mathrm{W}}(x) = \int \alpha(\Theta) e\left(-\tfrac{1}{2}\hbar \sum_{j<k} \varepsilon^{\alpha\beta} \theta_j{}^\alpha \theta_k{}^\beta\right) e\left(x^\alpha \sum_j \theta_j{}^\alpha\right) \mathrm{d}\Theta,$$

or

$$(\hat{A}^1 \ldots \hat{A}^n)_{\mathrm{W}}(x) = \left\{ e\left(\tfrac{1}{2}\hbar \sum_{j<k} \varepsilon^{\alpha\beta} \partial^2/\partial x_j^\alpha\, \partial x_k^\beta\right) A^1_{\mathrm{W}}(x_1) \ldots A^n_{\mathrm{W}}(x_n)\right\}\Big|_{x_1 = \ldots = x_n = x}. \tag{3.1}$$

The last line is most easily demonstrated by working backwards, expanding the exponential operator and using the definition of $\alpha(\Theta)$.

Equation (3.1) allows one to calculate the Wigner equivalent of a product from the Wigner equivalents of the multipliers. Groenewold's rule is obtained by setting $n = 2$:

$$[\hat{A}(\hat{x})\hat{B}(\hat{x})]_{\mathrm{W}} = \{e(\tfrac{1}{2}\hbar\varepsilon^{\alpha\beta}\, \partial^2/\partial x_1^\alpha\, \partial x_2^\beta) A_{\mathrm{W}}(x_1) B_{\mathrm{W}}(x_2)\}|_{x_1 = x_2 = x} \tag{3.2a}$$

$$= \{A_{\mathrm{W}}(x + \tfrac{1}{2}\mathrm{i}\hbar D_y) B_{\mathrm{W}}(y)\}|_{y=x} \tag{3.2b}$$

$$= \{B_{\mathrm{W}}(x - \tfrac{1}{2}\mathrm{i}\hbar D_y) A_{\mathrm{W}}(y)\}|_{y=x}, \tag{3.2c}$$

where $D_y^\alpha := \varepsilon^{\alpha\beta}\, \partial/\partial y_\beta$ (so that $D_x = (\nabla_p, -\nabla_q)$) and the equation $e(\tau \cdot \nabla_x) f(x) = f(x + \mathrm{i}\tau)$ has been used.

The most applicable case of (3.1) is when each $\hat{A}^i$ is a function of one phase space coordinate only , $\hat{x}^{\alpha_i}$ say. Then $[\hat{A}^i(\hat{x}^{\alpha_i})]_{\mathrm{W}} = A^i(x^{\alpha_i})$ and so one obtains

$$[\hat{A}^1(\hat{x}^{\alpha_1}) \ldots \hat{A}^n(\hat{x}^{\alpha_n})]_{\mathrm{W}} = \left\{ e\left(\tfrac{1}{2}\hbar \sum_{j<k} \varepsilon^{\alpha_j \alpha_k}\, \partial^2/\partial y_j\, \partial y_k\right) A^1(y_1) \ldots A^n(y_n)\right\}\Big|_{y_i = x^{\alpha_i}}. \tag{3.3}$$

Hence, if an operator $\hat{A}$ is expressed as a sum of products of this form, then its Wigner equivalent can be found from equation (3.3), bypassing the task of expressing $\hat{A}$ in normal form.

Finally, reverting to the usual phase space notation, if $\hat{A}(\hat{q}, \hat{p})$ is of the form $\hat{A}_1(\hat{q})\hat{B}_1(\hat{p}) \ldots \hat{A}_n(\hat{q})\hat{B}_n(\hat{p})$, then it may be shown from (3.1) that

$$A_{\mathrm{W}}(q, p) = \left\{ e\left(\tfrac{1}{2}\hbar\left[\sum_{i \leqslant j} \nabla q_i \cdot \nabla p_j - \sum_{i>j} \nabla q_i \cdot \nabla p_j\right]\right) A_1(q_1) B_1(p_1) \ldots A_n(q_n) B_n(p_n)\right\}\Big|_{\substack{q_i = q \\ p_i = p}}. \tag{3.4}$$

It can be seen that the sympletic notation is more compact and better suited to the expression of general forms.

4. Phase space multinomials

The results of the previous section will now be used to find the Wigner equivalent of the general multinomial $\hat{x}^{\alpha_1}\dots\hat{x}^{\alpha_n}$. From this the expansion of $\hat{x}^{\alpha_1}\dots\hat{x}^{\alpha_n}$ in terms of the symmetric multinomials of § 2 can be easily deduced.

From equation (3.3), putting $A^i(x)=x$, one obtains (where i, j, k range over $1,\dots,n$)

$$(\hat{x}^{\alpha_1}\dots\hat{x}^{\alpha_n})_{\mathrm{W}}=\left\{e\left(\tfrac{1}{2}\hbar\sum_{j<k}\varepsilon^{\alpha_j\alpha_k}\,\partial^2/\partial y_j\,\partial y_k\right)y_1\dots y_n\right\}\bigg|_{y_i=x^{\alpha_i}}$$

$$=x^{\alpha_1}\dots x^{\alpha_n}+\sum_{m=1}^{\infty}\frac{(\frac{1}{2}\mathrm{i}\hbar)^m}{m!}T_m(y)\bigg|_{y_i=x^{\alpha_i}} \tag{4.1}$$

from the expansion of $e(x)$, where one defines

$$T_m(y):=\left(\sum_{j<k}\varepsilon^{\alpha_j\alpha_k}\,\partial^2/\partial y_j\,\partial y_k\right)^m y_1\dots y_n$$

$$=\sum_{j_1<k_1}\dots\sum_{j_m<k_m}\varepsilon^{\alpha_{j_1}\alpha_{k_1}}\dots\varepsilon^{\alpha_{j_m}\alpha_{k_m}}\frac{\partial}{\partial y_{j_1}}\dots\frac{\partial}{\partial y_{k_m}}y_1\dots y_n.$$

Now,

$$\frac{\partial}{\partial y_{j_1}}\dots\frac{\partial}{\partial y_{k_m}}y_1\dots y_n$$

$$=\begin{cases}y_{\beta_1}\dots y_{\beta_{n-2m}} & \text{where } j_1,\dots,k_m \text{ are distinct and } 2m\leqslant n\\ 0 & \text{otherwise,}\end{cases}$$

where the sequence $\beta_1,\dots,\beta_{n-2m}$ is defined by removing $j_1,\dots,k_m$ from the sequence $1,\dots,n$.

Hence for $m>\lfloor\frac{1}{2}n\rfloor$ (where $\lfloor\frac{1}{2}n\rfloor$ denotes the integer part of $\frac{1}{2}n$) $T_m(y)\equiv 0$; otherwise, adopting the convention that where several variables appear under the one summation sign, the sum is to be evaluated over *distinct* values of these variables only, then one can write

$$T_m(y)=\sum_{\substack{j_1<k_1\\ \vdots\\ j_m<k_m}}\varepsilon^{\alpha_{j_1}\alpha_{k_1}}\dots\varepsilon^{\alpha_{j_m}\alpha_{k_m}}y_{\beta_1}\dots y_{\beta_{n-2m}}.$$

Noting that each term in the sum is independent of the ordering of the pairs $(j_1,k_1),\dots,(j_m,k_m)$, the result may be written more simply as

$$T_m(y)=m!\sum_{\substack{j_1<k_1\\ \vdots\\ j_m<k_m\\ (j_1<\dots<j_m)}}\varepsilon^{\alpha_{j_1}\alpha_{k_1}}\dots\varepsilon^{\alpha_{j_m}\alpha_{k_m}}y_{\beta_1}\dots y_{\beta_{n-2m}}.$$

Substitution of this expression into equation (4.1) yields the formula

$$(\hat{x}^{\alpha_1}\dots\hat{x}^{\alpha_n})_{\mathrm{W}}=x^{\alpha_1}\dots x^{\alpha_n}+\sum_{m=1}^{\lfloor n/2\rfloor}(\tfrac{1}{2}\mathrm{i}\hbar)^m\sum_{\substack{j_1<k_1\\ \vdots\\ j_m<k_m\\ (j_1<\dots<j_m)}}\varepsilon^{\alpha_{j_1}\alpha_{k_1}}\dots\varepsilon^{\alpha_{j_m}\alpha_{k_m}}x^{\beta_1}\dots x^{\beta_{n-2m}}, \tag{4.2}$$

where $\beta_1, \ldots, \beta_{n-2m}$ now denotes the sequence $\alpha_1, \ldots, \alpha_n$ with the terms $\alpha_{j_1}, \ldots, \alpha_{k_m}$ removed.

This gives an explicit expression for the Wigner equivalent of $\hat{x}^{\alpha_1} \ldots \hat{x}^{\alpha_n}$. For example, in the case $n=2$ one has $(\hat{x}^\alpha \hat{x}^\beta)_W = x^\alpha x^\beta + \frac{1}{2}i\hbar\varepsilon^{\alpha\beta}$. In particular $(\hat{q}\hat{p})_W = qp + \frac{1}{2}i\hbar$ while $(\hat{p}\hat{q})_W = pq - \frac{1}{2}i\hbar$.

It is possible to obtain equation (2.4) from (4.2) using $\varepsilon^{\alpha\beta} = -\varepsilon^{\beta\alpha}$. Thus either equation implies that

$$x^{\beta_1} \ldots x^{\beta_{n-2m}} = [\hat{x}^{(\beta_1} \ldots \hat{x}^{\beta_{n-2m})}]_W.$$

Substituting this into equation (4.2) and remembering that Weyl's rule is linear and one-one, we derive the operator relation

$$\hat{x}^{\alpha_1} \ldots \hat{x}^{\alpha_n} = \hat{x}^{(\alpha_1} \ldots \hat{x}^{\alpha_n)} + \sum_{m=1}^{[n/2]} (\tfrac{1}{2}i\hbar)^m \sum_{\substack{j_1<k_1 \\ \vdots \\ j_m<k_m \\ (j_1<\ldots<j_m)}} \varepsilon^{\alpha_{j_1}\alpha_{k_1}} \ldots \varepsilon^{\alpha_{j_m}\alpha_{k_m}} \hat{x}^{(\beta_1} \ldots \hat{x}^{\beta_{n-2m})}. \tag{4.3}$$

It is to be remembered that the $j_1, \ldots, k_m$ are to be summed over *distinct* values in the range $1, \ldots, n$. Hence, for example,

$$\hat{x}^\alpha \hat{x}^\beta \hat{x}^\gamma = \hat{x}^{(\alpha}\hat{x}^\beta \hat{x}^{\gamma)} + \tfrac{1}{2}i\hbar(\varepsilon^{\alpha\beta}\hat{x}^\gamma + \varepsilon^{\alpha\gamma}\hat{x}^\beta + \varepsilon^{\beta\gamma}\hat{x}^\alpha).$$

The symmetric multinomials are seen to form a complete basis for general multinomials; equation (4.3) gives the explicit expansion.

5. Discussion

The paper has mainly been concerned with the form Weyl's rule takes as a correspondence between the multinomial basis sets $\{x^{\alpha_1} \ldots x^{\alpha_n}\}$ and $\{\hat{x}^{(\alpha_1} \ldots \hat{x}^{\alpha_n)}\}$. This extends to more general functions as follows.

If equation (2.1) is multiplied by $e(-\theta^\alpha \xi^\alpha)$, where $\boldsymbol{\xi}$ is some point in phase space, then, writing $\boldsymbol{X} := \boldsymbol{x} - \boldsymbol{\xi}$, one obtains $e(\theta^\alpha X^\alpha) \leftrightarrow e(\theta^\alpha \hat{X}^\alpha)$. Thus Weyl's rule is invariant under translation, implying that $\boldsymbol{x}$ may be replaced by $\boldsymbol{X}$ in equations (2.4), (4.2) and (4.3). Hence, if the classical function $A(\boldsymbol{x})$ has a formal Taylor expansion about $\boldsymbol{\xi}$ of the form $A(\boldsymbol{x}) = \sum_{k=0}^{\infty} (1/k!) a_{\gamma_1 \ldots \gamma_k} X^{\gamma_1} \ldots X^{\gamma_k}$, then the corresponding operator induced by Weyl's rule is

$$\hat{A}(\hat{\boldsymbol{x}}) = \sum_{k=0}^{\infty} (1/k!) a_{\gamma_1 \ldots \gamma_k} \hat{X}^{(\gamma_1} \ldots \hat{X}^{\gamma_k)}.$$

Similarly, if going in the reverse direction an operator $\hat{A}(\hat{\boldsymbol{x}})$ has the multinomial expansion

$$\hat{A}(\hat{\boldsymbol{x}}) = \sum_{k=0}^{\infty} (1/k!) b_{\gamma_1 \ldots \gamma_k} \hat{X}^{\gamma_1} \ldots \hat{X}^{\gamma_k}, \tag{5.1}$$

then equation (4.2) with $\boldsymbol{X}$ substituted for $\boldsymbol{x}$ may be used to find $A_W(\boldsymbol{x})$ as power series in $\hbar$. Note that, while for a smooth function the $a_{\gamma_1 \ldots \gamma_k}$ coefficients are symmetric (i.e., $a_{\gamma_1 \ldots \gamma_k} = a_{(\gamma_1 \ldots \gamma_k)}$), this is not in general the case for the $b_{\gamma_1 \ldots \gamma_k}$. For example, take $\hat{A}(\hat{q}, \hat{p}) = \hat{q}\hat{p}$ and $\boldsymbol{\xi} = 0$. Then $b_{12} = 2$ and $b_{21} = 0$.

If $A(\boldsymbol{x})$ is a truly classical function in the sense that it is independent of $\hbar$, then the operator $\hat{A}$ related to A by $(\hat{A})_W \equiv A$ could be said to be *symmetrical*. Three equivalent conditions for $\hat{A}$ to be a symmetrical operator are that: $\partial A_W/\partial \hbar = 0$; the generalised Taylor series coefficients of equation (5.1) are symmetric, i.e., $b_{\gamma_1\ldots\gamma_k} = b_{(\gamma_1\ldots\gamma_k)}$; and finally $A_c(\boldsymbol{x}) = A_W(\boldsymbol{x})$, where $A_c(\boldsymbol{x})$ is the classical function obtained by replacing $\hat{\boldsymbol{x}}$ by $\boldsymbol{x}$ in the expression for $\hat{A}(\hat{\boldsymbol{x}})$ and letting $\hbar \to 0$.

The derivation of the equations in § 3 and § 4 relied essentially on three properties: that the components of $\boldsymbol{x}$ satisfy condition (A1) of the appendix and that Weyl's rule is both linear and one-one. Equation (4.3) can in fact be proven by algebraic manipulation of multinomial operators, independent of Weyl's rule and Wigner equivalents, but still relying on condition (A1). However, the proof is long (over 15 typewritten pages) and complex.

Correspondence rules of the form $e(\theta q + \tau p) \leftrightarrow f(\theta, \tau) e(\theta \hat{q} + \tau \hat{p})$ have been studied by Cohen (1966, 1976) in the context of general phase space distribution functions in quantum mechanics. It might be thought that the methods applied in this paper could be used to express multinomial operators in terms of basis functions other than symmetric multinomials. I have found, however, that the presence of $f(\theta, \tau)$ makes this impracticable.

It could seem that re-expressing results derived in the symplectic notation in terms of the usual phase space notation is difficult; this is not the case. If the symmetrised operator corresponding to $\hat{q}^m \hat{p}^n$ is to be denoted by $\hat{\Omega}_{mn}(\hat{q}, \hat{p})$, then $\hat{\Omega}_{mn}$ is given by the right-hand sides of equations (1.2*a*, *b*). Then in two dimensions for example, if some multinomial operator has a total of m_1 q_x's, m_2 q_y's, n_1 p_x's and n_2 p_y's, the commutation relations ensure that the corresponding symmetric multinomial is given by $\hat{\Omega}_{m_1 n_1}(\hat{q}_x, \hat{p}_x)\hat{\Omega}_{m_2 n_2}(\hat{q}_y, \hat{p}_y)$. Also the sums in equations (4.2) and (4.3) are quite easy to evaluate since $\varepsilon^{\alpha\beta}$ is mostly zero.

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Appendix

Here it is shown that

$$e^{\hat{A}_1} \ldots e^{\hat{A}_n} = e^{\frac{1}{2}\sum_{j<k}[\hat{A}_j, \hat{A}_k]} e^{\hat{A}_1 + \ldots + \hat{A}_n},$$

where the operators $\hat{A}_i$ satisfy the condition

$$[[\hat{A}_i, \hat{A}_j], \hat{A}_k] \equiv 0 \quad \forall\ i, j, k, \text{ in the range } 1, \ldots, n. \tag{A1}$$

For $n = 2$, the formula is well known (see e.g. Merzbacher 1970). The result for general n then follows easily by induction.

It should be noted that the components of $\hat{\boldsymbol{x}}$ satisfy condition (A1).

Note added in proof. The essential content of equation (2.4) is stated in § 2.4 of the review paper by N L Balazs and B K Jennings (1984 *Phys. Rep.* **104** 347).

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